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Representation theorems for generators of backward stochastic differential equations and their applications ☆

Long Jiang[a,b,*]

[a]*Department of Mathematics, China University of Mining and Technology, Xuzhou 221008, PR China*

[b]*School of Mathematics and System Sciences, Shandong University, Jinan 250100, PR China*

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Abstract

We prove that the generator g of a backward stochastic differential equation (BSDE) can be represented by the solutions of the corresponding BSDEs at point (t, y, z) if and only if t is a conditional Lebesgue point of generator g with parameters (y, z). By this conclusion, we prove that, if g is a Lebesgue generator and g is independent of y, then, Jensen's inequality for g-expectation holds if and only if g is super homogeneous; we also obtain a converse comparison theorem for deterministic generators of BSDEs.

MSC: primary 60H10

Keywords: Backward stochastic differential equation; Representation theorem; Conditional Lebesgue point; Lebesgue generator; g-Expectation; Converse comparison theorem

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*Corresponding author at: Department of Mathematics, China University of Mining and Technology, Xuzhou 221008, PR China.

E-mail address: jianglong365@hotmail.com.

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1. Introduction

It is by now well-known (see [10]) that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE) of type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T, \tag{1}$$

providing, for instance, that the function g is Lipschitz in both variables y and z, and that ξ and $(g(t, 0, 0))_{t\in[0,T]}$ are square integrable. g is said to be the generator of BSDE (1). We denote the unique adapted and square integrable solution of BSDE (1) by $(Y_t(g, T, \xi), Z_t(g, T, \xi))_{t\in[0,T]}$. When g also satisfies $g(t, y, 0) \equiv 0$ for any (t, y), then, $Y_0(g, T, \xi)$, denoted by $\mathscr{E}_g[\xi]$, is called g-expectation of ξ; $Y_t(g, T, \xi)$, denoted by $\mathscr{E}_g[\xi|\mathscr{F}_t]$, is called conditional g-expectation of ξ (see [11]).

Since this kind of BSDEs and the notion of g-expectation was introduced, many properties of BSDEs and g-expectations have been studied. Some papers [1,3,4], have been devoted to Jensen's inequality for g-expectation. Roughly speaking, the problem of Jensen's inequality for g-expectation is: for convex function $\varphi : \mathbf{R} \mapsto \mathbf{R}$, what conditions should be given to the generator g such that the following inequality:

$$\mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t] \geqslant \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)]$$

will hold in general?

One of the achievements of BSDEs theory is the Comparison Theorem, some papers [1,2,5,8], have been devoted to a kind of converse comparison problem, their main problem is

If two generators g_1 and g_2 satisfy $\mathscr{E}_{g_1}[\xi] \geqslant \mathscr{E}_{g_2}[\xi]$ for any ξ, can we prove that $g_1 \geqslant g_2$?

For studying this kind of converse comparison problem, Briand et al. [1] established the following representation theorem: $\forall (t, y, z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d$,

$$L^2 - g(t, y, z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)) - y] \tag{2}$$

for generator g under two additional assumptions that $(g(t, y, z))_{t\in[0,T]}$ is continuous with respect to t for each (y, z) and $\mathbf{E}[\sup_{0\leqslant t\leqslant T} |g(t, 0, 0)|^2] < \infty$.

The first objective of this paper is to establish a more generalized Representation Theorem for generators of BSDEs. Then, applying this new Representation Theorem, this paper gives an answer to Jensen's inequality for g-expectation and it also obtains a converse comparison theorem for deterministic generators.

The remainder of this paper is organized in four Sections. In Section 2, we will introduce the notations, assumptions, definitions and some lemmas which will be useful in this paper; in Section 3, we establish our new Representation Theorem, we will prove that the generator g of a BSDE can be represented at point (t, y, z), i.e., equality (2) holds at point (t, y, z), if and only if t is a conditional Lebesgue point of g with parameters (y, z); we also introduce three kinds of Lebesgue generators in this Section; in Section 4, we prove that if g is a Lebesgue generator and g is independent of y, then, Jensen's inequality holds for g-expectation if and only if g is super-homogeneous; in Section 5, we will establish a converse comparison theorem for deterministic generators.

2. Preliminaries

In this section, we introduce some notations, assumptions, definitions and lemmas which will be useful in this paper.

Let $T>0$ be a given real number; let $(\Omega, \mathscr{F}, P)$ be a probability space and $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$; let $(\mathscr{F}_t)_{t\geqslant 0}$ be the filtration generated by this Brownian motion:

$$\mathscr{F}_t = \sigma\{B_s, s \in [0, t]\} \vee \mathscr{N}, \quad t \in [0, T],$$

where $\mathscr{N}$ is the set of all P-null subsets. For any positive integers m, n and $z = (z_{ij})_{m\times n} \in \mathbf{R}^{m\times n}$, $|z| := (\sum_{i,j} z_{ij}^2)^{\frac{1}{2}}$.

We define the following usual spaces of processes:

$$\mathscr{S}^2_{\mathscr{F}}(0, T; \mathbf{R}) := \{\psi \text{ continuous and progressively measurable}; \mathbf{E}[\sup_{0\leqslant t\leqslant T} |\psi_t|^2] < \infty\};$$
$$\mathscr{H}^2_{\mathscr{F}}(0, T; \mathbf{R}^n) := \{\psi \text{ progressively measurable}; \|\psi\|_2^2 = \mathbf{E}[\textstyle\int_0^T |\psi_t|^2\, \mathrm{d}t] < \infty\}.$$

The generator g of a BSDE is a function $g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \longmapsto \mathbf{R}$ such that $(g(t, y, z))_{t\in[0,T]}$ is progressively measurable for each $(y, z) \in \mathbf{R} \times \mathbf{R}^d$, and g also satisfies the following standard assumptions (A1) and (A2):

(A1) (Lipschitz Condition) There exists a constant $K \geqslant 0$, such that P-a.s., we have: $\forall t, \forall y_1, y_2, z_1, z_2 : \ |g(t, y_1, z_1) - g(t, y_2, z_2)| \leqslant K(|y_1 - y_2| + |z_1 - z_2|)$.
(A2) The process $(g(t, 0, 0))_{t\in[0,T]} \in \mathscr{H}^2_{\mathscr{F}}(0, T; \mathbf{R})$.
(A3) P-a.s. $\forall (t, y) \in [0, T] \times \mathbf{R}$, $g(t, y, 0) \equiv 0$.

Let (A1) and (A2) hold for g. Then for each $\xi \in L^2(\Omega, \mathscr{F}_T, P)$, by Pardoux and Peng [10], BSDE (1) has a unique adapted solution, denoted by $(Y_t(g, T, \xi), Z_t(g, T, \xi))_{t\in[0,T]}$, in $\mathscr{S}^2_{\mathscr{F}}(0, T; \mathbf{R}) \times \mathscr{H}^2_{\mathscr{F}}(0, T; \mathbf{R}^d)$.

For the convenience of readers, we list two Lemmas which comes from [1].

Lemma 2.1 (*Representation Theorem*). *Let Assumptions* (A1) *and* (A2) *hold for* g. *Let* $(g(t, y, z))_{t\in[0,T]}$ *be continuous with respect to* t *for each* (y, z) *and* $\mathbf{E}[\sup_{0\leqslant t\leqslant T} |g(t, 0, 0)|^2] < \infty$. *Let* $b : \mathbf{R}^n \longmapsto \mathbf{R}^n, \sigma : \mathbf{R}^n \longmapsto \mathbf{R}^{n\times d}$ *be two Lipschitz functions. Let* $0 \leqslant t < T$; *we denote by* $X^{t,x}$ *the solution of* SDE:

$$X_s^{t,x} = x + \int_t^s b(X_u^{t,x})\, \mathrm{d}u + \int_t^s \sigma(X_u^{t,x}) \cdot \mathrm{d}B_u, \quad s \in [t, T], \ \ X_s^{t,x} = x \quad \textit{if} \ s \in [0, t].$$

Then for each $(t, x, y, q) \in [0, T[\times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, *we have*

$$L^2 - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + q \cdot (X^{t,x}_{t+\varepsilon} - x)) - y] = g(t, y, \sigma^*(x)q) + q \cdot b(x).$$

Lemma 2.2. *Let Assumptions* (A1) *and* (A2) *hold for* g, *and let* $\xi \in L^2(\Omega, \mathscr{F}_T, P)$. *Then the solution* $(y_t, z_t)_{t\in[0,T]}$ *of* BSDE (1) *satisfies*

$$\mathbf{E}\left[\sup_{t\leqslant s\leqslant T}(\mathrm{e}^{\beta s}|y_s|^2) + \int_t^T \mathrm{e}^{\beta s}|z_s|^2\,\mathrm{d}s|\mathscr{F}_t\right]$$
$$\leqslant C\mathbf{E}\left[\mathrm{e}^{\beta T}|\xi|^2 + \left(\int_t^T \mathrm{e}^{(\beta/2)s}|g(s,0,0)|\,\mathrm{d}s\right)^2 |\mathscr{F}_t\right],$$

where $\beta = 2(K + K^2)$ *and* C *is a universal constant.*

We recall the notion of g-expectation and conditional g-expectation defined by Peng [11], we also list some basic properties of g-expectation which were obtained by Peng [11]. In the following Definitions 2.1, 2.2 and Lemmas 2.3–2.6, g is assumed to satisfy (A1) and (A3).

Definition 2.1. The g-expectation $\mathscr{E}_g[\cdot] : L^2(\Omega, \mathscr{F}_T, P) \longmapsto \mathbf{R}$ is defined by

$$\mathscr{E}_g[\xi] := Y_0(g, T, \xi).$$

Definition 2.2. The conditional g-expectation of ξ with respect to $\mathscr{F}_t$ is defined by

$$\mathscr{E}_g[\xi|\mathscr{F}_t] := Y_t(g, T, \xi).$$

Lemma 2.3. (1) (*Preserving of constants*): *For each constant* $c, \mathscr{E}_g[c] = c$;

(2) (*Monotonicity*): *If* $X_1 \geqslant X_2$, a.s., *then* $\mathscr{E}_g[X_1] \geqslant \mathscr{E}_g[X_2]$;
(3) (*Strict Monotonicity*): *If* $X_1 \geqslant X_2$, a.s., *and* $P(X_1 > X_2) > 0$, *then* $\mathscr{E}_g[X_1] > \mathscr{E}_g[X_2]$.

Lemma 2.4. (1) *If* X *is* $\mathscr{F}_t$*-measurable, then* $\mathscr{E}_g[X|\mathscr{F}_t] = X$;
(2) *For all* $t, s \in [0, T], \mathscr{E}_g[\mathscr{E}_g[X|\mathscr{F}_t]|\mathscr{F}_s] = \mathscr{E}_g[X|\mathscr{F}_{t\wedge s}]$.
(3) *For each* $t \in [0, T], \mathscr{E}_g[\mathscr{E}_g[X|\mathscr{F}_t]] = \mathscr{E}_g[X]$.

Lemma 2.5. $\mathscr{E}_g[X|\mathscr{F}_t]$ *is the unique random variable* η *in* $L^2(\Omega, \mathscr{F}_t, P)$, *such that*

$$\mathscr{E}_g[X1_A] = \mathscr{E}_g[\eta 1_A] \quad \text{for all } A \in \mathscr{F}_t.$$

Lemma 2.6. *Let generator* g *satisfy* (A1) *and* (A3). *If* g *does not depend on* y, *i.e.*, $g(\omega, t, z)\colon \Omega \times [0, T] \times \mathbf{R}^d \longmapsto \mathbf{R}$. *Then*

$$\mathscr{E}_g[X + \eta|\mathscr{F}_t] = \mathscr{E}_g[X|\mathscr{F}_t] + \eta \quad \forall \eta \in L^2(\Omega, \mathscr{F}_t, P), \;\; X \in L^2(\Omega, \mathscr{F}_T, P).$$

3. Representation theorems for BSDEs

Before we state and prove our new Representation Theorem, we first establish a more general Lemma.

Let $b(\cdot,\cdot,\cdot):\Omega\times[0,T]\times\mathbf{R}^n\longmapsto\mathbf{R}^n$, $\sigma(\cdot,\cdot,\cdot):\Omega\times[0,T]\times\mathbf{R}^n\longmapsto\mathbf{R}^{n\times d}$ be two functions such that for any $x\in\mathbf{R}^n$, $b(\cdot,\cdot,x)$ and $\sigma(\cdot,\cdot,x)$ are both progressively measurable; let b and σ also satisfy the following hypotheses (H1)–(H3):

(H1) Lipschitz condition: There exists a constant $K_1\geqslant 0$ such that

$$|b(t,x)-b(t,y)|+|\sigma(t,x)-\sigma(t,y)|\leqslant K_1|x-y|\quad \forall x,y\in\mathbf{R}^n,\ \ t\in[0,T],$$

(H2) linear growth condition: There exists a constant $K_2\geqslant 0$ such that

$$|b(t,x)|+|\sigma(t,x)|\leqslant K_2(1+|x|)\quad \forall x\in\mathbf{R}^n,\ \ t\in[0,T],$$

(H3) *for each* $x\in\mathbf{R}^n$, $t\mapsto b(t,x)$, $t\mapsto\sigma(t,x)$ *are both right continuous in* $t\in[0,T[$.

Given $(t,x)\in[0,T[\times\mathbf{R}^n$, we denote by $X^{t,x}$ the solution of the following SDE (3):

$$X_s^{t,x}=x+\int_t^s b(u,X_u^{t,x})\,\mathrm{d}u+\int_t^s\sigma(u,X_u^{t,x})\cdot\mathrm{d}B_u,\quad s\in[t,T],\ \ X_s^{t,x}=x,\ \ s\in[0,t]. \tag{3}$$

Then, by the classical SDE theory (see Theorem 5.1 of Chapter 1 of Ref. [9] *for details*) we understand that the above SDE (3) has a unique s-continuous solution $(X_s^{t,x})_{s\in[0,T]}$ with properties that $(X_s^{t,x})_{s\in[0,T]}$ is $(\mathscr{F}_s)$-adapted,

$$\mathbf{E}\left[\sup_{0\leqslant s\leqslant T}|X_s^{t,x}|^2\right]<\infty, \tag{4}$$

and

$$s\longmapsto\mathbf{E}[|X_s^{t,x}-x|]^2,\quad s\in[0,T],\quad \text{is continuous.} \tag{5}$$

Given $(t,x,y,q)\in[0,T[\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$. We choose $\varepsilon>0$ small enough such that $t+\varepsilon\leqslant T$. Let (A1) and (A2) hold for generator g, then the following BSDE

$$Y_s^\varepsilon=y+q\cdot(X_{t+\varepsilon}^{t,x}-x)+\int_s^{t+\varepsilon}g(u,Y_u^\varepsilon,Z_u^\varepsilon)\,\mathrm{d}u-\int_s^{t+\varepsilon}Z_u^\varepsilon\cdot\mathrm{d}B_u,\quad s\in[0,t+\varepsilon] \tag{6}$$

has a unique solution in $\mathscr{S}^2_{\mathscr{F}}(0,t+\varepsilon;\mathbf{R})\times\mathscr{H}^2_{\mathscr{F}}(0,t+\varepsilon;\mathbf{R}^d)$, we denote it by

$$(Y_s(g,t+\varepsilon,y+q\cdot(X_{t+\varepsilon}^{t,x}-x)),Z_s(g,t+\varepsilon,y+q\cdot(X_{t+\varepsilon}^{t,x}-x)))_{s\in[0.t+\varepsilon]}.$$

Then, motivated by Lemma 2.1, we have

Lemma 3.1. *Let* (A1) *and* (A2) *hold for generator* g; *let* (H1)–(H3) *hold for* b *and* σ; *let* $1\leqslant p\leqslant 2$. *Then for each* $(t,x,y,q)\in[0,T[\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$, *the following two statements are equivalent*:

(i) $L^p-g(t,y,\sigma^*(t,x)q)+q\cdot b(t,x)=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}[Y_t(g,t+\varepsilon,y+q\cdot(X_{t+\varepsilon}^{t,x}-x))-y]$;
(ii) $L^p-g(t,y,\sigma^*(t,x)q)=\lim_{\varepsilon\to 0^+}\mathbf{E}[\frac{1}{\varepsilon}\int_t^{t+\varepsilon}g(u,y,\sigma^*(t,x)q)\,\mathrm{d}u|\mathscr{F}_t]$.

Proof. Given $(t,x,y,q)\in[0,T[\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n$. For notations simplicity we denote by $(Y_s^\varepsilon,Z_s^\varepsilon)_{s\in[0,t+\varepsilon]}$ the solution of BSDE (6). For $s\in[t,t+\varepsilon]$, we set

$$\tilde{Y}_s^\varepsilon := Y_s^\varepsilon-(y+q\cdot(X_s^{t,x}-x)) \quad\text{and}\quad \tilde{Z}_s^\varepsilon := Z_s^\varepsilon-\sigma^*(s,X_s^{t,x})q.$$

Then, for $s\in[t,t+\varepsilon]$, applying Itô's formula to $\tilde{Y}_s^\varepsilon$, we have

$$\tilde{Y}_s^\varepsilon = \int_s^{t+\varepsilon}[g(u,\tilde{Y}_u^\varepsilon+y+q\cdot(X_u^{t,x}-x),\tilde{Z}_u^\varepsilon+\sigma^*(u,X_u^{t,x})q)+q\cdot b(u,X_u^{t,x})]\,\mathrm{d}u \\ -\int_s^{t+\varepsilon}\tilde{Z}_u^\varepsilon\cdot\mathrm{d}B_u. \tag{7}$$

Then we have the following Proposition 3.2.

Proposition 3.2. $\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\mathbf{E}[\sup_{t\leqslant s\leqslant t+\varepsilon}|\tilde{Y}_s^\varepsilon|^2+\int_t^{t+\varepsilon}|\tilde{Z}_s^\varepsilon|^2\,\mathrm{d}s]=0.$

Proof. By Lemma 2.2, Lipschitz condition (A1), (H1) and Hölder's inequality we know there exists a universal constant C such that

$$\begin{aligned}
&\mathbf{E}\left[\sup_{t\leqslant s\leqslant t+\varepsilon}|\tilde{Y}_s^\varepsilon|^2+\int_t^{t+\varepsilon}|\tilde{Z}_u^\varepsilon|^2\,\mathrm{d}u\;|\mathscr{F}_t\right]\\
&\leqslant C\mathrm{e}^{2(K+K^2)\mathrm{T}}\mathbf{E}\left[\left(\int_t^{t+\varepsilon}|g(u,y+q\cdot(X_u^{t,x}-x),\sigma^*(u,X_u^{t,x})q)\right.\right.\\
&\qquad\left.\left.+q\cdot b(u,X_u^{t,x})|\,\mathrm{d}u\right)^2\;|\mathscr{F}_t\right]\\
&\leqslant C\mathrm{e}^{2(K+K^2)\mathrm{T}}\mathbf{E}\left[\left(\int_t^{t+\varepsilon}(|g(u,y,\sigma^*(u,x)q)|+(K+KK_1+K_1)|q||X_u^{t,x}-x|\right.\right.\\
&\qquad\left.\left.+|q\cdot b(u,x)|)\,\mathrm{d}u\right)^2\;|\mathscr{F}_t\right]\\
&\leqslant C_1\mathbf{E}\left[\varepsilon\int_t^{t+\varepsilon}(|g(u,y,\sigma^*(u,x)q)|+|q||X_u^{t,x}-x|+|q\cdot b(u,x)|)^2\,\mathrm{d}u\;|\mathscr{F}_t\right]\\
&\leqslant 4C_1\varepsilon\mathbf{E}\left[\int_t^{t+\varepsilon}(|g(u,y,\sigma^*(u,x)q)|^2+|q|^2|X_u^{t,x}-x|^2+|q\cdot b(u,x)|^2)\,\mathrm{d}u\;|\mathscr{F}_t\right],
\end{aligned}$$

where $C_1 := C\mathrm{e}^{2(K+K^2)\mathrm{T}}(1+K+KK_1+K_1)^2$ is a positive constant. Thus we have

$$\begin{aligned}
&\mathbf{E}\left[\sup_{t\leqslant s\leqslant t+\varepsilon}|\tilde{Y}_s^\varepsilon|^2+\int_t^{t+\varepsilon}|\tilde{Z}_u^\varepsilon|^2\,\mathrm{d}u\right]\\
&\leqslant 4C_1\varepsilon\mathbf{E}\left[\int_t^{t+\varepsilon}(|g(u,y,\sigma^*(u,x)q)|^2+|q|^2|X_u^{t,x}-x|^2+|q\cdot b(u,x)|^2)\,\mathrm{d}u\right].
\end{aligned}$$

By (H2) we have

$$\mathbf{E}\left[\int_t^{t+\varepsilon} |q \cdot b(u,x)|^2 \,\mathrm{d}u\right] \leqslant \int_t^{t+\varepsilon} |q|^2 \mathbf{E}[|b(u,x)|^2]\,\mathrm{d}u$$
$$\leqslant \int_t^{t+\varepsilon} |q|^2 K_2^2 (1+|x|)^2 \,\mathrm{d}u \to 0 \quad (\varepsilon \to 0^+). \tag{8}$$

By Fubini Theorem and (4), we have

$$\mathbf{E}\left[\int_t^{t+\varepsilon} |q|^2 |X_u^{t,x} - x|^2 \,\mathrm{d}u\right] \leqslant \int_t^{t+\varepsilon} |q|^2 2\mathbf{E}(|X_u^{t,x}|^2 + |x|^2)\,\mathrm{d}u \to 0 \quad (\varepsilon \to 0^+). \tag{9}$$

By Lipschitz condition (A1) and (H2), we have

$$|g(u,y,\sigma^*(u,x)q)|^2 \leqslant 2|g(u,y,\sigma^*(t,x)q)|^2 + 2K^2|\sigma^*(u,x)q - \sigma^*(t,x)q|^2$$
$$\leqslant 2|g(u,y,\sigma^*(t,x)q)|^2 + 8K^2K_2^2(1+|x|)^2|q|^2.$$

Thanks to (A2) and the absolute continuity of integral, we have

$$\lim_{\varepsilon \to 0^+} \mathbf{E}\left[\int_t^{t+\varepsilon} |g(u,y,\sigma^*(t,x)q)|^2 \,\mathrm{d}u\right] = 0.$$

Thus we also have

$$\lim_{\varepsilon \to 0^+} \mathbf{E}\left[\int_t^{t+\varepsilon} |g(u,y,\sigma^*(u,x)q)|^2 \,\mathrm{d}u\right] = 0. \tag{10}$$

Thus Proposition 3.2 follows from (8)–(10). □

We set

$$M_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} g(u, \tilde{Y}_u^\varepsilon + y + q\cdot(X_u^{t,x} - x), \tilde{Z}_u^\varepsilon + \sigma^*(u, X_u^{t,x})q)\,\mathrm{d}u|\mathscr{F}_t\right],$$
$$N_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} g(u, y + q\cdot(X_u^{t,x} - x), \sigma^*(u, X_u^{t,x})q)\,\mathrm{d}u|\mathscr{F}_t\right],$$
$$P_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} g(u,y,\sigma^*(u,x)q)\,\mathrm{d}u\ |\mathscr{F}_t\right],$$
$$Q_t^\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} g(u,y,\sigma^*(t,x)q)\,\mathrm{d}u\ |\mathscr{F}_t\right].$$

Taking conditional expectation with respect to $\mathscr{F}_t$ in BSDE (7), we get

$$\frac{1}{\varepsilon}(Y_t^\varepsilon - y) = \frac{1}{\varepsilon}\ \tilde{Y}_t^\varepsilon = M_t^\varepsilon + \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u, X_u^{t,x})\,\mathrm{d}u\ |\mathscr{F}_t\right]. \tag{11}$$

Thus we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}(Y_t^\varepsilon - y) - [g(t,y,\sigma^*(t,x)q) + q\cdot b(t,x)] \\
&\quad = [M_t^\varepsilon - g(t,y,\sigma^*(t,x)q)] + \left[\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u,X_u^{t,x})\,\mathrm{d}u\;|\mathscr{F}_t\right] - q\cdot b(t,x)\right] \\
&= (M_t^\varepsilon - N_t^\varepsilon) + (N_t^\varepsilon - P_t^\varepsilon) + (P_t^\varepsilon - Q_t^\varepsilon) \\
&\quad + \left[\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u,X_u^{t,x})\,\mathrm{d}u\,|\mathscr{F}_t\right] - q\cdot b(t,x)\right] \\
&\quad + \left[\mathbf{E}\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(u,y,\sigma^*(t,x)q)\,\mathrm{d}u|\mathscr{F}_t\right) - g(t,y,\sigma^*(t,x)q)\right].
\end{aligned}
$$

By Jensen's inequality, Hölder's inequality and Lipschitz condition (H1) we conclude

$$
\begin{aligned}
&\mathbf{E}\left|\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u,X_u^{t,x})\,\mathrm{d}u\;|\mathscr{F}_t\right] - q\cdot b(t,x)\right|^2 \\
&\quad = \mathbf{E}\left|\frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot (b(u,X_u^{t,x}) - b(u,x))\,\mathrm{d}u\,|\mathscr{F}_t\right] + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} q\cdot (b(u,x)-b(t,x))\,\mathrm{d}u\right|^2 \\
&\quad \leqslant \frac{2}{\varepsilon}\,\mathbf{E}\left[\int_t^{t+\varepsilon} |q|^2K_1^2|X_u^{t,x}-x|^2\,\mathrm{d}u\right] + \mathbf{E}\left[\frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2|b(u,x)-b(t,x)|^2\,\mathrm{d}u\right].
\end{aligned}
$$

Noticing that $\mathbf{E}[|X_t^{t,x}-x|^2]=0$, then by (5) we get

$$
\lim_{\varepsilon\to 0^+}\frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2K_1^2\mathbf{E}[|X_u^{t,x}-x|^2]\,\mathrm{d}u = 0. \tag{12}
$$

Since $b(u,x)$ is right continuous with respect to u for each x, thus

$$
\lim_{\varepsilon\to 0^+}\frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2|b(u,x)-b(t,x)|^2\,\mathrm{d}u = 0, \quad P\text{-a.s.}
$$

Since for each given x, b is bounded, then, by Lebesgue's dominated convergence theorem we have

$$
\lim_{\varepsilon\to 0^+}\mathbf{E}\left[\frac{2}{\varepsilon}\int_t^{t+\varepsilon}|q|^2|b(u,x)-b(t,x)|^2\,\mathrm{d}u\right] = 0. \tag{13}
$$

Therefore,

$$
L^2-\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\,\mathbf{E}\left[\int_t^{t+\varepsilon} q\cdot b(u,X_u^{t,x})\,\mathrm{d}u\;|\mathscr{F}_t\right] = q\cdot b(t,x). \tag{14}
$$

By Jensen's inequality, Hölder's inequality, Lipschitz condition (A1) and Proposition 3.2 we deduce

$$
\mathbf{E}[M_t^\varepsilon - N_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} 2K^2(|\tilde{Y}_u^\varepsilon|^2 + |\tilde{Z}_u^\varepsilon|^2)\,\mathrm{d}u\right] \to 0 \quad (\varepsilon\to 0^+). \tag{15}
$$

By Jensen's inequality, Hölder's inequality, Lipschitz condition (A1) and (H1) we conclude

$$\mathbf{E}[N_t^\varepsilon - P_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\,\mathbf{E}\left[\int_t^{t+\varepsilon} K^2|q|^2(1+K_1)^2|X_u^{t,x} - x|^2\,\mathrm{d}u\right],$$

it follows from Fubini Theorem and (12) that

$$\mathbf{E}[N_t^\varepsilon - P_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon} K^2|q|^2(1+K_1)^2\mathbf{E}|X_u^{t,x} - x|^2\,\mathrm{d}u \to 0 \quad (\varepsilon \to 0^+). \tag{16}$$

By Jensen's inequality, Hölder's inequality and Lipschitz condition (A1) we conclude

$$\mathbf{E}[P_t^\varepsilon - Q_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\,\mathbf{E}\left[\int_t^{t+\varepsilon} K^2|q|^2|\sigma^*(u,x) - \sigma^*(t,x)|^2\,\mathrm{d}u\right].$$

Since $\sigma(u,x)$ is right continuous with respect to u for each x, thus

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} K^2|q|^2|\sigma^*(u,x) - \sigma^*(t,x)|^2\,\mathrm{d}u \to 0, \quad P\text{-a.s.}$$

Then, it follows from (H2) and Lebesgue's dominated convergence theorem that

$$\mathbf{E}[P_t^\varepsilon - Q_t^\varepsilon]^2 \leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon} K^2|q|^2\mathbf{E}[|\sigma^*(u,x) - \sigma^*(t,x)|^2]\,\mathrm{d}u \to 0 \quad (\varepsilon \to 0^+). \tag{17}$$

Thus Lemma 3.1 follows from (14)–(17). $\square$

From Lemma 3.1, we can get the following Representation Theorem for generators of BSDEs immediately.

Theorem 3.3 (*Representation Theorem*). *Let* (A1) *and* (A2) *hold for* g; *let* $1\leqslant p\leqslant 2$. *Then for each triplet* $(t,y,z)\in[0,T[\times\mathbf{R}\times\mathbf{R}^d$, *the following two statements are equivalent*:

(i) $L^p - g(t,y,z) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}[Y_t(g,t+\varepsilon, y + z\cdot(B_{t+\varepsilon} - B_t)) - y]$;
(ii) $L^p - g(t,y,z) = \lim_{\varepsilon\to 0^+} \mathbf{E}[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s,y,z)\,\mathrm{d}s|\mathscr{F}_t]$.

Remark 3.1. Obviously, we can get similar convergence results in probability sense.

Definition 3.1. Let (A1) and (A2) hold for generator g; let $1\leqslant p\leqslant 2$.

(i) Let $(y,z)\in\mathbf{R}\times\mathbf{R}^d$ and $t\in[0,T[$, if

$$L^p - g(t,y,z) = \lim_{\varepsilon\to 0^+} \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s,y,z)\,\mathrm{d}s|\mathscr{F}_t\right],$$

then we say t is a conditional Lebesgue point of g with parameters (y,z) (in L^p sense).

(ii) If for any $(y,z) \in \mathbf{R} \times \mathbf{R}^d$,

$$L^p - g(t,y,z) = \lim_{\varepsilon \to 0^+} \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s,y,z)\,\mathrm{d}s|\mathscr{F}_t\right]$$

holds for almost every $t \in [0,T[$. Then, we say g is a Lebesgue generator (in L^p sense).

Remark 3.2. We can define conditional Lebesgue point and Lebesgue generator in probability sense similarly.

In the following of this section, we introduce three kinds of Lebesgue generators of BSDEs. We first conclude that if g is a deterministic generator, i.e., $g(t,y,z) : [0,T] \times \mathbf{R} \times \mathbf{R}^d \longmapsto \mathbf{R}$, then g is a Lebesgue generator. Indeed, we have

Theorem 3.4. *Let g be deterministic and* (A1) *and* (A2) *hold for g. Then*

(i) *g is a Lebesgue generator in L^2 sense,*
(ii) *for each pair $(y,z) \in \mathbf{R} \times \mathbf{R}^d$, we have*

$$g(t,y,z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\,[Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)) - y], \quad \text{a.e., } \mathrm{d}t;$$

(iii) *moreover, if g also satisfies* (A3), *then for each pair $(y,z) \in \mathbf{R} \times \mathbf{R}^d$,*

$$g(t,y,z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\,[\mathscr{E}_g[y + z \cdot (B_{t+\varepsilon} - B_t)] - y], \quad \text{a.e., } \mathrm{d}t.$$

Proof. (i) Given $(y,z) \in \mathbf{R} \times \mathbf{R}^d$. Since (A2) holds for g and g is deterministic, then, by Hewitt and Stromberg [7, Lemma 18.4] we know that almost every $t \in [0,T]$ is Lebesgue point of function $g(\cdot,y,z)$. Thus for each given pair (y,z),

$$g(t,y,z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s,y,z)\,\mathrm{d}s, \quad \text{a.e., } \mathrm{d}t. \tag{18}$$

Since g is deterministic, (i) does hold.

(ii) Since g is deterministic, it follows from Proposition 4.2 of El Karoui et al. [6] that $Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t))$ is deterministic. Thus (ii) follows from (18) and Theorem 3.3.

(iii) Suppose moreover that (A3) also holds for g. Then, by Lemmas 2.3, 2.4 and that $Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t))$ is deterministic we infer

$$\begin{aligned}
&\mathscr{E}_g[y + z \cdot (B_{t+\varepsilon} - B_t)] \\
&\quad = \mathscr{E}_g[\mathscr{E}_g[y + z \cdot (B_{t+\varepsilon} - B_t)|\mathscr{F}_t]] \\
&\quad = \mathscr{E}_g[Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t))] \\
&\quad = Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)).
\end{aligned}$$

Thus (iii) follows (ii). $\square$

If g is right continuous with respect to t, then we have

Proposition 3.5. *Let* (A1), (A2) *and the following two additional Assumptions* (A4) *and* (A5) *hold for* g:

(A4) P-a.s., $\forall (y,z) \in \mathbf{R} \times \mathbf{R}^d$, $t \longmapsto g(t,y,z)$ *is right continuous in* $t \in [0,T[$;

(A5) *For any* $t \in [0,T[$, $E[|g(t,0,0)|^2] < \infty$, *and there exist two positive constants* δ_t *and* K_t, *such that*

$$E\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g^2(s,0,0)\,\mathrm{d}s\right] < K_t \quad \forall\ 0<\varepsilon\leqslant \min\{\delta_t,\ T-t\}.$$

Then for any $p \in [1,2[$, g *is a Lebesgue generator in* L^p *sense*; *in fact*, *for each triplet* $(t,y,z) \in [0,T[\times \mathbf{R} \times \mathbf{R}^d$,

$$L^p - g(t,y,z) = \lim_{\varepsilon\to 0^+} \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s,y,z)\,\mathrm{d}s|\mathscr{F}_t\right].$$

Proof. Given $t \in [0,T[$, given $(y,z) \in \mathbf{R} \times \mathbf{R}^d$. Let $0<\varepsilon<T-t$. We set

$$H_\varepsilon := \frac{1}{\varepsilon}\mathbf{E}\left[\int_t^{t+\varepsilon} [g(s,y,z) - g(t,y,z)]\,\mathrm{d}s|\mathscr{F}_t\right],$$

then for any $p \in [1,2[$, applying Jensen's inequality we can get that

$$\mathbf{E}[|H_\varepsilon|^p] \leqslant \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|\,\mathrm{d}s\right]^p. \tag{19}$$

Due to the right continuity of $g(t,y,z)$ with respect to $t \in [0,T[$, we conclude that

$$P\text{-a.s.,}\quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|\,\mathrm{d}s = 0. \tag{20}$$

On the other hand, from Hölder's inequality and the Lipschitz condition we can get that

$$\begin{aligned}
\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|\,\mathrm{d}s\right]^2 &\leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|^2\,\mathrm{d}s \\
&\leqslant \frac{1}{\varepsilon}\int_t^{t+\varepsilon} [|g(s,y,z) - g(s,0,0)| \\
&\quad + |g(t,y,z) - g(t,0,0)| + |g(s,0,0) - g(t,0,0)|]^2\,\mathrm{d}s \\
&\leqslant \frac{4}{\varepsilon}\int_t^{t+\varepsilon} 4K^2(|y|^2+|z|^2)\,\mathrm{d}s + \frac{4}{\varepsilon}\int_t^{t+\varepsilon} 2(|g(s,0,0)|^2 + |g(t,0,0)|^2)\,\mathrm{d}s \\
&= 16K^2(|y|^2+|z|^2) + \frac{8}{\varepsilon}\int_t^{t+\varepsilon} |g(s,0,0)|^2\,\mathrm{d}s + 8|g(t,0,0)|^2.
\end{aligned}$$

For the given $t \in [0, T[$, by (A5), we know that $E[|g(t,0,0)|^2] < \infty$ and there exist $\delta_t > 0$ small enough and a constant $K_t > 0$ such that $t + \delta_t \leqslant T$ and

$$\mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g^2(s,0,0)\,\mathrm{d}s\right] < K_t \quad \forall\ 0 < \varepsilon \leqslant \min\{\delta_t,\ T-t\}.$$

Therefore we have

$$\begin{aligned}&\mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|\,\mathrm{d}s\right]^2 \\ &\quad \leqslant 16K^2(|y|^2 + |z|^2) + 8\mathbf{E}[|g(t,0,0)|^2] + 8K_t \quad \forall\ 0 < \varepsilon \leqslant \min\{\delta_t,\ T-t\}.\end{aligned}$$

Thus for any $p \in [1, 2[$, combining the above inequality with (20) we conclude that

$$\lim_{\varepsilon \to 0^+} \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|\,\mathrm{d}s\right]^p = 0.$$

Thus by (19) we have that

$$\lim_{\varepsilon \to 0^+} \mathbf{E}[|H_\varepsilon|^p] \leqslant \lim_{\varepsilon \to 0^+} \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |g(s,y,z) - g(t,y,z)|\,\mathrm{d}s\right]^p = 0.$$

The proof of Proposition 3.5 is complete. $\square$

By Lebesgue's dominated convergence theorem, we can get the following Proposition 3.6 immediately.

Proposition 3.6. *Let* (A1) *and* (A4) *hold for* g. *Suppose* $E[\sup_{t \in [0,T]} |g(t,0,0)|^2] < \infty$. *Then* g *is a Lebesgue generator in* L^2 *sense*; *in fact for each triplet* $(t,y,z) \in [0, T[\times \mathbf{R} \times \mathbf{R}^d$,

$$L^2 - g(t,y,z) = \lim_{\varepsilon \to 0^+} \mathbf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(s,y,z)\,\mathrm{d}s|\mathscr{F}_t\right].$$

4. Jensen's inequality for g-expectation

In this section, we always consider the situation where the generator g does not depend on y, that is, $g : \Omega \times [0, T] \times \mathbf{R}^d \longmapsto \mathbf{R}$. We always assume that g satisfies (A1) and (A3).

Definition 4.1. Let g satisfy (A1) and (A3). We say that g is a super-homogeneous generator if for each $(\lambda, z) \in \mathbf{R} \times \mathbf{R}^d$, g also satisfies:

$$g(t, \lambda z) \geqslant \lambda g(t,z), \quad \text{a.s., a.e.}$$

Now we introduce some results on Jensen's inequality for g-expectation.

Theorem 4.1. *Let* (A1) *and* (A3) *hold for* g; *let* $1 \leqslant p \leqslant 2$ *and* g *is a Lebesgue generator in* L^p *sense. Then the following two statements are equivalent*:

(i) g *is a super-homogeneous generator*;
(ii) *Jensen's inequality for* g*-expectation holds in general, i.e., for each* $\xi \in L^2(\Omega, \mathscr{F}_T, P)$ *and convex function* $\varphi : \mathbf{R} \to \mathbf{R}$, *if* $\varphi(\xi) \in L^2(\Omega, \mathscr{F}_T, P)$, *then*

$$P\text{-a.s.} \quad \forall t \in [0, T], \quad \mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t] \geqslant \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)].$$

Proof. (i) $\Rightarrow$ (ii): Given $\xi \in L^2(\Omega, \mathscr{F}_T, P)$ and convex function φ such that $\varphi(\xi) \in L^2(\Omega, \mathscr{F}_T, P)$. Given $t \in [0, T]$, we set $\eta_t = \varphi'_-[\mathscr{E}_g(\xi|\mathscr{F}_t)]$, then η_t is $\mathscr{F}_t$-measurable. Since φ is convex, we have

$$\varphi(x) - \varphi(y) \geqslant \varphi'_-(y)(x - y) \quad \forall\, x, \;\; y \in R.$$

Take $x = \xi, y = \mathscr{E}_g(\xi|\mathscr{F}_t)$, then we have

$$\varphi(\xi) - \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)] \geqslant \eta_t[\xi - \mathscr{E}_g(\xi|\mathscr{F}_t)].$$

For each positive integer n, we define $\Omega_{t,n} := \{|\mathscr{E}_g(\xi|\mathscr{F}_t)| + |\eta_t| + |\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)]| \leqslant n\}$. Because $\mathscr{E}_g[\xi|\mathscr{F}_t], \eta_t, \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)]$ are all $\mathscr{F}_t$-measurable, so $\Omega_{t,n} \in \mathscr{F}_t$. We denote the indicator function of $\Omega_{t,n}$ by $\mathbf{1}_{\Omega_{t,n}}$, set $\eta_{t,n} = \mathbf{1}_{\Omega_{t,n}}\eta_t$. Then we have

$$\mathbf{1}_{\Omega_{t,n}}[\varphi(\xi) - \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)]] \geqslant \eta_{t,n}[\xi - \mathscr{E}_g(\xi|\mathscr{F}_t)]. \tag{21}$$

Since $\eta_{t,n}$, $\mathbf{1}_{\Omega_{t,n}}\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)]$ are bounded by n and $\xi, \varphi(\xi) \in L^2(\Omega, \mathscr{F}_T, P)$, we deduce that

$$\mathbf{1}_{\Omega_{t,n}}\varphi(\xi),\ \eta_{t,n}\xi \in L^2(\Omega, \mathscr{F}_T, P),\ \mathbf{1}_{\Omega_{t,n}}\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)] \in L^2(\Omega, \mathscr{F}_t, P),$$
$$(\eta_{t,n}\mathscr{E}_g(\xi|\mathscr{F}_s))_{t \leqslant s \leqslant T} \in S^2_{\mathscr{F}}(t, T; R).$$

From the well-known Comparison Theorem we know that conditional g-expectation $\mathscr{E}_g[\cdot|\mathscr{F}_t]$ is nondecreasing. Thus from inequality (21), by taking conditional g-expectation, we can get

$$\mathscr{E}_g[\mathbf{1}_{\Omega_{t,n}}[\varphi(\xi) - \varphi(\mathscr{E}_g(\xi|\mathscr{F}_t))]|\mathscr{F}_t] \geqslant \mathscr{E}_g[\eta_{t,n}[\xi - \mathscr{E}_g(\xi|\mathscr{F}_t)]|\mathscr{F}_t].$$

Since $\mathbf{1}_{\Omega_{t,n}}\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)], \eta_{t,n}\mathscr{E}_g[\xi|\mathscr{F}_t] \in L^2(\Omega, \mathscr{F}_t, P)$ and g is independent of y, it follows from Lemma 2.6 that

$$\mathscr{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathscr{F}_t] - \mathbf{1}_{\Omega_{t,n}}\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)] \geqslant \mathscr{E}_g[\eta_{t,n}\xi|\mathscr{F}_t] - \eta_{t,n}\mathscr{E}_g[\xi|\mathscr{F}_t]. \tag{22}$$

Let $(y_u, z_u)_{u \in [0,T]}$ be the solution of the following BSDE (23):

$$y_u = \xi + \int_u^T g(s, z_s)\,\mathrm{d}s - \int_u^T z_s \cdot \mathrm{d}B_s, \quad 0 \leqslant u \leqslant T. \tag{23}$$

Then for the given $t \in [0, T]$, we have

$$\eta_{t,n}y_u = \eta_{t,n}\xi + \int_u^T \eta_{t,n}g(s, z_s)\,\mathrm{d}s - \int_t^T \eta_{t,n}z_s \cdot \mathrm{d}B_s, \quad t \leqslant u \leqslant T. \tag{24}$$

We define function $g_1(s,z)$ in this way: for each $(s,z)\in[t,T]\times\mathbf{R}^d$,

$$g_1(s,z) := \begin{cases} \eta_{t,n} g(s, z/\eta_{t,n}) & \text{if } \eta_{t,n}\neq 0, \\ 0 & \text{if } \eta_{t,n}=0. \end{cases}$$

Since $\eta_{t,n}$ is bounded, the following BSDE,

$$\bar{y}_u = \eta_{t,n}\xi + \int_u^T g_1(s,\bar{z}_s)\,\mathrm{d}s - \int_t^T \bar{z}_s\cdot \mathrm{d}B_s, \quad t\leqslant u\leqslant T, \tag{25}$$

has a unique solution in $\mathscr{S}^2_{\mathscr{F}}(t,T;\mathbf{R})\times\mathscr{H}^2_{\mathscr{F}}(t,T;\mathbf{R}^d)$. We denote it by $(\bar{y}_s,\bar{z}_s)_{s\in[t,T]}$. Also from the fact that $\eta_{t,n}$ is bounded we know that $(\eta_{t,n}y_s,\eta_{t,n}z_s)_{s\in[t,T]}$ is in $\mathscr{S}^2_{\mathscr{F}}(t,T;\mathbf{R})\times\mathscr{H}^2_{\mathscr{F}}(t,T;\mathbf{R}^d)$. From (24) and the definition of g_1, we conclude that the solution of BSDE (25) is just $(\eta_{t,n}y_s,\eta_{t,n}z_s)_{s\in[t,T]}$.

Consider the solutions of BSDE (25) and the following BSDE (26):

$$\tilde{y}_u = \eta_{t,n}\xi + \int_u^T g(s,\tilde{z}_s)\,\mathrm{d}s - \int_u^T \tilde{z}_s\cdot \mathrm{d}B_s, \quad t\leqslant u\leqslant T. \tag{26}$$

Due to the super-homogeneity of g, we deduce that

$$g(s,\eta_{t,n}z_s)\geqslant \eta_{t,n}g(s,z_s), \quad \text{a.s., a.e. on } \Omega\times[t,T].$$

Combining this with the definition of g_1, we have

$$\begin{aligned} g(s,\bar{z}_s) &= g(s,\eta_{t,n}z_s)\geqslant \eta_{t,n}g(s,z_s) = g_1(s,\eta_{t,n}z_s) \\ &= g_1(s,\bar{z}_s), \quad \text{a.s., a.e. on } \Omega\times[t,T]. \end{aligned}$$

Thus from Comparison Theorem we have

$$P\text{-a.s.,}\quad \mathscr{E}_g[\eta_{t,n}\xi|\mathscr{F}_t] = \tilde{y}_t \geqslant \bar{y}_t = \eta_{t,n}y_t = \eta_{t,n}\mathscr{E}_g[\xi|\mathscr{F}_t]. \tag{27}$$

Coming back to (22), we can get

$$\mathscr{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathscr{F}_t] - \mathbf{1}_{\Omega_{t,n}}\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)] \geqslant \mathscr{E}_g[\eta_{t,n}\xi|\mathscr{F}_t] - \eta_{t,n}\mathscr{E}_g[\xi|\mathscr{F}_t]\geqslant 0. \tag{28}$$

Applying Lebesgue's dominated convergence theorem to $(\mathbf{1}_{\Omega_{t,n}}\varphi(\xi))_{n=1}^{\infty}$, we can get easily that

$$L^2 - \lim_{n\to\infty} \mathbf{1}_{\Omega_{t,n}}\varphi(\xi) = \varphi(\xi).$$

Since that $\xi\to\mathscr{E}_g(\xi|\mathscr{F}_t)$ is a continuous map from $L^2(\mathscr{F}_T)$ into $L^2(\mathscr{F}_t)$ (see [11, Lemma 36.9]), it follows that

$$L^2 - \lim_{n\to\infty} \mathscr{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathscr{F}_t] = \mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t]. \tag{29}$$

On the other hand, by the definition of $\Omega_{t,n}$, we can get, P-a.s.,

$$\lim_{n\to\infty} \mathbf{1}_{\Omega_{t,n}}\varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)] = \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)]. \tag{30}$$

It follows from (29), (30) and (28) that

$$\mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t] \geqslant \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)], \quad P\text{-a.s.}$$

Hence (i) implies (ii) indeed.

(ii) $\Rightarrow$ (i): For each $\lambda \in \mathbf{R}, z \in \mathbf{R}^d$, $t \in [0, T[$. Suppose that t is a conditional Lebesgue point of g with parameter z and also t is a conditional Lebesgue point of g with parameter λz in L^p sense. Then, by Theorem 3.3, we have

$$L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathscr{E}_g[z \cdot (B_{t+\varepsilon} - B_t)|\mathscr{F}_t] = g(t, z), \tag{31}$$

$$L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathscr{E}_g[\lambda z \cdot (B_{t+\varepsilon} - B_t)|\mathscr{F}_t] = g(t, \lambda z). \tag{32}$$

For the given $\lambda \in \mathbf{R}$, we define a corresponding convex function $\varphi_\lambda : \mathbf{R} \mapsto \mathbf{R}$, such that $\varphi_\lambda(x) = \lambda x$, $\forall x \in \mathbf{R}$. For the given $t \in [0, T[$, let us pick a large enough positive integer n such that $t + 1/n \leqslant T$. Then for the given $z \in \mathbf{R}^d$, it is obvious that $\varphi_\lambda(z \cdot (B_{t+1/n} - B_t)) \in L^2(\Omega, \mathscr{F}_T, P)$. By (ii), we know that, P-a.s.,

$$\mathscr{E}_g[\varphi_\lambda(z \cdot (B_{t+1/n} - B_t))|\mathscr{F}_t] \geqslant \varphi_\lambda[\mathscr{E}_g(z \cdot (B_{t+1/n} - B_t)|\mathscr{F}_t)],$$

that is, P-a.s.,

$$\mathscr{E}_g[\lambda z \cdot (B_{t+1/n} - B_t)|\mathscr{F}_t] \geqslant \lambda[\mathscr{E}_g(z \cdot (B_{t+1/n} - B_t)|\mathscr{F}_t)]. \tag{33}$$

Thus for the given $\lambda \in \mathbf{R}, z \in \mathbf{R}^d, t \in [0, T[$, by (31)–(33), we have P-a.s.,

$$g(t, \lambda z) \geqslant \lambda g(t, z). \tag{34}$$

Since g is a Lebesgue generator in L^p sense, then for each $\lambda \in \mathbf{R}, z \in \mathbf{R}^d$, (31) and (32) hold for almost every $t \in [0, T[$. Thus for each given $(\lambda, z) \in \mathbf{R} \times \mathbf{R}^d$, it follows from (34) that

$$g(t, \lambda z) \geqslant \lambda g(t, z), \quad \text{a.s., a.e.}$$

Therefore (ii) implies (i). The proof is complete. □

Definition 4.2. We call a generator g is positive homogeneous if for each $(\lambda, z) \in \mathbf{R}_+ \times \mathbf{R}^d$,

$$g(t, \lambda z) = \lambda g(t, z), \quad \text{a.s., a.e.}$$

Theorem 4.2. *Let* (A1) *and* (A3) *hold for g; let $1 \leqslant p \leqslant 2$ and g be a Lebesgue generator in L^p sense. Suppose for each $t \in \mathbf{R}$, P-a.s., $z \longmapsto g(t, z)$ is convex in z. Then the following two statements are equivalent*:

(i) *g is positive homogeneous*,
(ii) *Jensen's inequality for g-expectation holds in general.*

Proof. By Theorem 4.1, it suffices to prove that if g is convex in z and $g(t, 0) \equiv 0$, then g is positive homogeneous if and only if g is super-homogeneous.

Suppose g is positive homogeneous. We only need to consider the case when $\lambda \leqslant 0$. For each $\lambda \leqslant 0$, $z \in \mathbf{R}^d$, since g is convex and $g(t, 0) \equiv 0$, we have

$$\begin{aligned} 0 = g(t,0) &= g\left(t, \frac{\lambda z}{2} + \frac{(-\lambda)z}{2}\right) \\ &\leqslant \frac{g(t,\lambda z)}{2} + \frac{g(t,-\lambda z)}{2} \\ &= \frac{g(t,\lambda z)}{2} + \frac{-\lambda g(t,z)}{2}, \quad \text{a.s., a.e.} \end{aligned}$$

Thus for each $\lambda \leqslant 0$, $z \in \mathbf{R}^d$, we have

$$g(t, \lambda z) \geqslant \lambda g(t,z), \quad \text{a.s., a.e.}$$

Hence g is super-homogeneous.

Suppose g is super-homogeneous. For each pair $(\lambda, z) \in [0,1] \times \mathbf{R}^d$, by the convexity of g and (A3) we have

$$g(t, \lambda z) \leqslant \lambda g(t,z), \quad \text{a.s., a.e.}$$

Thus for pair $(\lambda, z) \in [0,1] \times \mathbf{R}^d$, by the super-homogeneity of g, we have

$$g(t, \lambda z) = \lambda g(t,z), \quad \text{a.s., a.e.} \tag{35}$$

For pair $(\lambda, z) \in [1, +\infty[\times \mathbf{R}^d$, it follows from (35) that

$$\lambda g(t,z) = \lambda g\left(t, \frac{1}{\lambda} \times (\lambda z)\right) = \lambda \times \frac{1}{\lambda} \times g(t, \lambda z) = g(t, \lambda z), \quad \text{a.s., a.e..}$$

Thus g is positive homogeneous. The proof is complete. $\square$

The following Example 4.1 will show that a super-homogeneous generator which satisfies (A1) and (A3) is not necessarily convex.

Example 4.1. Take $d = 2$. Define a function $g = g(z) : \mathbf{R}^2 \longmapsto \mathbf{R}$, such that for $\forall z = (u, v)^{\mathrm{T}} \in \mathbf{R}^2$

$$\begin{aligned} g(z) :=& \max\{\min\{u, -v\}, v - u\} \\ =& \begin{cases} u & \text{if } z \in S_1, \\ -v & \text{if } z \in S_2, \\ v - u & \text{if } z \in S_3 \cup S_4, \end{cases} \end{aligned}$$

where

$$S_1 := \{z = (u,v)^{\mathrm{T}},\ v \leqslant 2u,\ v \leqslant -u\}, \quad S_2 := \{z = (u,v)^{\mathrm{T}},\ -2u \leqslant 2v \leqslant u\},$$
$$S_3 := \{z = (u,v)^{\mathrm{T}};\ u \leqslant 2v\}, \quad S_4 := \{z = (u,v)^{\mathrm{T}};\ 2u \leqslant v\}.$$

Then obviously that $g(0) = 0$ and g is positive homogeneous. Now let us verify that g is Lipschitz, super-homogeneous and g is not convex.

First, let us verify that this generator g is Lipschitz. Let $z_i = (u_i, v_i)^{\mathrm{T}} \in S_i$, $i = 1, 2, 3, 4$. Then

$$\begin{aligned}
&u_1 - u_2 \leqslant g(z_1) - g(z_2) = u_1 + v_2 \leqslant v_2 - v_1,\\
&v_1 - u_1 + u_3 - v_3 \leqslant g(z_1) - g(z_3) = u_1 + u_3 - v_3 \leqslant u_1 + v_3 \leqslant v_3 - v_1,\\
&v_1 - u_1 + u_4 - v_4 \leqslant g(z_1) - g(z_4) = u_1 + u_4 - v_4 \leqslant u_1 - u_4,\\
&v_2 - u_2 + u_3 - v_3 \leqslant g(z_2) - g(z_3) = -v_2 + u_3 - v_3 \leqslant -v_2 + v_3,\\
&v_2 - u_2 + u_4 - v_4 \leqslant g(z_2) - g(z_4) = -v_2 + u_4 - v_4 \leqslant u_2 - u_4,\\
&\qquad\qquad\qquad |g(z_3) - g(z_4)| = |v_3 - v_4 + u_4 - u_3| \leqslant 2|z_3 - z_4|.
\end{aligned}$$

Therefore g satisfies Lipschitz condition. Indeed, we have

$$|g(z_i) - g(z_j)| \leqslant 2|z_i - z_j|, \quad i,j = 1, 2, 3, 4.$$

Second, Let's verify that g is super-homogeneous. Obviously that g is positive homogeneous, so we only need to prove that for $\forall\, z \in \mathbf{R}^2, \lambda \leqslant 0$,

$$g(\lambda z) \geqslant \lambda g(z).$$

For any $z = (u, v)^{\mathrm{T}} \in \mathbf{R}^2$, if $z \in S_1$, then $-z = (-u, -v)^{\mathrm{T}} \in S_3 \cup S_4$. Thus

$$g(z) + g(-z) = u + (-v + u) = 2u - v \geqslant 0.$$

If $z = (u, v)^{\mathrm{T}} \in S_2$, then $-z = (-u, -v)^{\mathrm{T}} \in S_3 \cup S_4$. Thus we have

$$g(z) + g(-z) = -v + (-v + u) = u - 2v \geqslant 0.$$

If $z = (u, v)^{\mathrm{T}} \in S_3 \cap S_4$, then $-z = (-u, -v)^{\mathrm{T}} \in S_1 \cup S_2$. Thus by the above two inequalities we also have

$$g(z) + g(-z) \geqslant 0.$$

If $z = (u, v)^{\mathrm{T}} \in S_3 \backslash S_4$, then $-z = (-u, -v)^{\mathrm{T}} \in S_4 \backslash S_3$. Thus

$$g(z) + g(-z) = 0.$$

If $z = (u, v)^{\mathrm{T}} \in S_4 \backslash S_3$, then $-z = (-u, -v)^{\mathrm{T}} \in S_3 \backslash S_4$. Thus we also have

$$g(z) + g(-z) = 0.$$

Therefore for $\forall\, z \in \mathbf{R}^2$, we have

$$g(-z) \geqslant -g(z).$$

Since g is positive homogeneous, then for $\forall\, z \in \mathbf{R}^2, \lambda \leqslant 0$, we have

$$g(\lambda z) = g((-\lambda)(-z)) = (-\lambda) g(-z) \geqslant (-\lambda)(-g(z)) = \lambda g(z).$$

Therefore g is super-homogeneous indeed.

Third, we prove that g is not convex and also g is not concave. Indeed, if we take $z_1 = (-1, -5)^{\mathrm{T}}, z_2 = (1, 0)^{\mathrm{T}},\ z_3 = (1, 3)^{\mathrm{T}}, \alpha = \frac{1}{2}$, then

$$g(\alpha z_1 + (1 - \alpha) z_2) = 0 > -\tfrac{1}{2} = \alpha g(z_1) + (1 - \alpha) g(z_2),$$

$$g(\alpha z_1 + (1 - \alpha) z_3) = 0 < \tfrac{1}{2} = \alpha g(z_1) + (1 - \alpha) g(z_3).$$

Thus g is not convex and also g is not concave. Thus g satisfies (A1) and (A3), g is super-homogeneous, but g is not convex.

For monotonic convex function, analogous to the proof of Theorem 4.1, we can obtain the following Theorems 4.3 and 4.4.

Theorem 4.3. *Let* (A1) *and* (A3) *hold for* g; *let* $1 \leqslant p \leqslant 2$ *and* g *is a Lebesgue generator in* L^p *sense. Then the following two statements are equivalent*:

(i) *for each pair* $(\lambda, z) \in \mathbf{R}_+ \times \mathbf{R}^d$, $g(t, \lambda z) \geqslant \lambda g(t, z)$, a.s., a.e.,
(ii) *Jensen's inequality for* g*-expectation holds for increasing convex function, i.e., for each* $\xi \in L^2(\Omega, \mathscr{F}_T, P)$ *and increasing convex function* $\varphi : \mathbf{R} \to \mathbf{R}$, *if* $\varphi(\xi) \in L^2(\Omega, \mathscr{F}_T, P)$, *then*

$$P\text{-a.s.} \quad \forall t \in [0, T], \qquad \mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t] \geqslant \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)].$$

Theorem 4.4. *Let* (A1) *and* (A3) *hold for* g; *let* $1 \leqslant p \leqslant 2$ *and* g *is a Lebesgue generator in* L^p *sense. Then the following two statements are equivalent*:

(i) *for each* $z \in \mathbf{R}^d, \lambda \leqslant 0$, $g(t, \lambda z) \geqslant \lambda g(t, z)$, a.s., a.e.,
(ii) *Jensen's inequality for* g*-expectation holds for decreasing convex function, i.e., for each* $\xi \in L^2(\Omega, \mathscr{F}_T, P)$ *and decreasing convex function* $\varphi : \mathbf{R} \mapsto \mathbf{R}$, *if* $\varphi(\xi) \in L^2(\Omega, \mathscr{F}_T, P)$, *then*

$$P\text{-a.s.} \quad \forall t \in [0, T], \qquad \mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t] \geqslant \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)].$$

Remark 4.1. In the proof of Theorems 4.1–4.4, when we prove that (i) implies (ii), we do not need to assume that g is a Lebesgue generator.

From Theorem 4.3, we can get the following Corollary 4.5 immediately:

Corollary 4.5. *Given* $\mu \geqslant 0$, *let generator* $g(z) = -\mu|z|$, $\forall z \in \mathbf{R}^d$. *Then Jensen's inequality for* g*-expectation holds for increasing convex function* φ.

By Theorem 4.4 and Remark 4.1, we can obtain the following Corollary 4.6 immediately:

Corollary 4.6. *Let* (A1) *and* (A3) *hold for* g. *If* $g \geqslant 0$, *then Jensen's inequality for* g*- expectation holds for decreasing convex function* φ.

By Theorem 4.2 and Corollary 4.6, we can construct an example easily to show that Jensen's inequality for g-expectation does not hold in general.

Example 4.2. Let $g : \mathbf{R} \mapsto \mathbf{R}$ be defined as follows: $g(z) = z^4$, if $|z| \leqslant 1$ and $g(z) = 4|z| - 3$, if $|z| > 1$. We can see clearly that though g is convex, g is not positive homogeneous. Thus for this generator g, by Theorem 4.2, we know that Jensen's inequality for g-expectation does not hold in general. Since $g \geqslant 0$, then, by Corollary 4.6, we know that Jensen's inequality for g-expectation holds for decreasing convex functions.

Let us take $T=1$, $\xi = B_T - T$ and $\varphi(x) = \frac{x}{3}$, $\forall x \in \mathbf{R}$, then we can verify that $(B_t - t, 1)_{t\in[0,T]}$ is the solution of the following BSDE:

$$y_t = \xi + \int_t^T g(z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T,$$

and $(\frac{B_t}{3} - \frac{26T+t}{81}, \frac{1}{3})_{t\in[0,T]}$ is the solution of the following BSDE:

$$\bar{y}_t = \varphi(\xi) + \int_t^T g(\bar{z}_s)\,\mathrm{d}s - \int_t^T \bar{z}_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T.$$

We can calculate that

$$\mathscr{E}_g[\varphi(\xi)|\mathscr{F}_t] - \varphi[\mathscr{E}_g(\xi|\mathscr{F}_t)] = \tfrac{26}{81}(t-T) < 0, \quad \text{when } t < T.$$

5. A converse comparison theorem for BSDEs

In this short section, we will establish a converse comparison theorem for generators of BSDEs. We all know that one of the achievements of BSDEs theory is the Comparison Theorem, which is due to El Karoui [6]. Some papers [1,2,5,8] have been devoted to converse comparison theorem for g-expectations. Coquet et al. [5] solves this problem under the additional assumption that the two generators are both continuous with respect to t.

Since the main results obtained on this topic are all dependent on the assumption that the generators are continuous with respect to t, then, a natural question is asked:

If the generators are not continuous with respect to t, can we establish a converse comparison theorem?

Thanks to Theorem 3.4, we can establish a converse comparison theorem for deterministic generators without the continuity assumption.

Theorem 5.1. *Let two generators g_1 and g_2 satisfy Assumptions* (A1) *and* (A3) *and be both deterministic. Then the following three statements are equivalent*:

(i) *for each pair $(y,z) \in \mathbf{R}\times\mathbf{R}^d$, we have*: $g_1(t,y,z) \geqslant g_2(t,y,z)$, *a.e.*,
(ii) *for each $\xi \in L^2(\Omega, \mathscr{F}_T, P)$, we have*: $\mathscr{E}_{g_1}[\xi] \geqslant \mathscr{E}_{g_2}[\xi]$;
(iii) *for each pair $(y,z) \in \mathbf{R}\times\mathbf{R}^d$ and for each $t \in [0,T[, \varepsilon \in]0, T-t]$, we have*

$$\mathscr{E}_{g_1}[y + z\cdot(B_{t+\varepsilon} - B_t)] \geqslant \mathscr{E}_{g_2}[y + z\cdot(B_{t+\varepsilon} - B_t)].$$

Proof. It follows from the well-known Comparison Theorem that (i) implies (ii); it is obvious that (ii) implies (iii). Now let us prove that (iii) implies (i). Because g_1 and g_2 are both deterministic, thus for each pair $(y,z) \in \mathbf{R}\times\mathbf{R}^d$, by Theroem 3.4 we know that

$$g_1(t,y,z) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\,[\mathscr{E}_{g_1}[y + z\cdot(B_{t+\varepsilon} - B_t)] - y], \quad \text{a.e., } \mathrm{d}t.$$

$$g_2(t,y,z) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\,[\mathscr{E}_{g_2}[y + z\cdot(B_{t+\varepsilon} - B_t)] - y], \quad \text{a.e., d}t.$$

Therefore (iii) implies (i) indeed. The proof of Theorem 5.1 is complete. □

From Theorem 5.1, we can get the following Corollary 5.2 immediately.

Corollary 5.2. *Let two generators g_1 and g_2 satisfy Assumptions* (A1) *and* (A3) *and be both deterministic. Then the following three statements are equivalent*:

(i) *for each pair* $(y,z)\in \mathbf{R}\times\mathbf{R}^d$, *we have*: $g_1(t,y,z) = g_2(t,y,z)$, a.e.,
(ii) *for each*, $\xi\in L^2(\Omega,\mathscr{F}_T,P)$, *we have*: $\mathscr{E}_{g_1}[\xi] = \mathscr{E}_{g_2}[\xi]$,
(iii) *for each pair* $(y,z)\in\mathbf{R}\times\mathbf{R}^d$ *and for each* $t\in[0,T[, \varepsilon\in]0,T-t]$, *we have*

$$\mathscr{E}_{g_1}[y + z\cdot(B_{t+\varepsilon} - B_t)] = \mathscr{E}_{g_2}[y + z\cdot(B_{t+\varepsilon} - B_t)].$$

Readers may wonder if we weaken Assumption (A3) on generators g_1 and g_2, for example, g_1 satisfies Assumption (A3) and g_2 satisfies the following Assumption (A6):

(A6) P-a.s., $\forall t\in[0,T] \quad g(t,0,0)\equiv 0$,
then, can we obtain a converse comparison theorem similar to Theorem 5.1? Generally the answer is negative. Please study the following Example 5.1.

Example 5.1. Let us define two generators of BSDEs

$$g_1(t,y,z) :\equiv 0, \quad g_2(t,y,z) := (2t-T)y, \quad \forall (t,y,z)\in[0,T]\times\mathbf{R}\times\mathbf{R}^d.$$

Then obviously that g_1 satisfies (A3) and g_2 satisfies (A6), but g_2 does not satisfy (A3).

For any $\xi\in Ł^2(\Omega,\mathscr{F}_T,P)$, let $(Y_t^2, Z_t^2)_{t\in[0,T]}$ denote the solution of the following BSDE (36):

$$Y_t^2 = \xi + \int_t^T g_2(s, Y_s^2, Z_s^2)\,\mathrm{d}s - \int_t^T Z_s^2\cdot \mathrm{d}B_s, \quad 0\leqslant t\leqslant T. \tag{36}$$

Applying Itô's formula to $Y_t^2\exp(\int_0^t (2s-T)\,\mathrm{d}s)$, we can get that

$$\begin{aligned} d&\left[Y_t^2\exp\left(\int_0^t (2s-T)\,\mathrm{d}s\right)\right] \\ &= \exp\left(\int_0^t (2s-T)\,\mathrm{d}s\right)\mathrm{d}Y_t^2 + Y_t^2 d\left[\exp\left(\int_0^t (2s-T)\,\mathrm{d}s\right)\right] \\ &= \left[\exp\left(\int_0^t (2s-T)\,\mathrm{d}s\right)\right] Z_t^2\cdot \mathrm{d}B_t, \quad \forall\ t\in[0,T]. \end{aligned}$$

Integrate over the interval $[0,T]$, then, by taking expectation, we conclude that

$$Y_0^2 = \mathbf{E}\left[Y_T^2\exp\left(\int_0^T (2s-T)\,\mathrm{d}s\right) - \int_0^T \left(\exp\left(\int_0^t (2s-T)\,\mathrm{d}s\right)\right) Z_t^2\cdot \mathrm{d}B_t\right]$$

$$= \mathbf{E}[Y_T^2] = \mathbf{E}[\xi]$$
$$= \mathscr{E}_{g_1}[\xi].$$

But obviously the inequality $g_1 \geqslant g_2$ does not hold.

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